ON THE STABILITY OF THE SOLUTIONS OF A SYSTEM OF HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC (AND OTHER) COEFFICIENTS

(OB USTOICHIVOSTI RESHENII SISTEMY LINEINYKH ODNORODNYKH DIFFERENTSIAL'NYKH URAVNENII S PERIODICHESKIMI (I DRUGIMI) KOEFFITSIYENTAMI)

PMM Vol.23, No.5, 1959, pp. 818-825

N.P. ERUGIN
(Minsk)

(Received 10 June 1959)

This article presents an asymptotic method for the determination of the eigenvalues of certain systems of homogeneous linear differential equations, and techniques for the determination of the exponential matrix. The radius of convergence is indicated for the series which represent the invariants of the exponential matrix.

1. Let the n-order matrix

$$V(\lambda) = \sum_{k=0}^{\infty} \lambda^k V_k \tag{1.1}$$

be given, where λ is a numerical variable, the matrices V_k are independent of λ, and the series (1.1) converges in the region

$$|\lambda| < r \tag{1.2}$$

We will assume

$$W = \ln V(\lambda) = \sum_{k=0}^{\infty} \lambda^k W_k \tag{1.3}$$

and that the series (1.3) converges in the region $|\lambda| \leqslant r_1 < r$.

Theorem 1. If the eigenvalues of the matrix $V(\lambda)$ do not vanish in the region (1.2), then the invariants of the matrix W have a series representation in integral powers of λ, converging in the region (1.2).

Proof. The eigenvalues $v_1(\lambda), \ldots, v_n(\lambda)$ of the matrix $V(\lambda)$ are determined from the equation

$$v^n + \overline{V}_1(\lambda)\, v^{n+1} + \cdots + \overline{V}_n(\lambda) = 0 \qquad (1.4)$$

where $V_k(\lambda)$ $(k = 1, \ldots, n)$ are holomorphic functions of λ in the region (1.2). It follows that in the region (1.2) the eigenvalues $v_k(\lambda)$ have only algebraic singularities, and that for any λ_0 from the region (1.2), the eigenvalues can be represented in the form of series in integral powers of $(\lambda - \lambda_0)$ or $(\lambda - \lambda_0)^{1/k}$, where k is a positive integer, with $k < n$. The eigenvalues $w_1, \ldots, w_n$ of the matrix W are equal [1] to $\ln v_1(\lambda), \ldots, \ln v_n(\lambda)$ (we consider them as the principal values [2]).

The invariants σ_k of the matrix W are symmetric polynomials in $\ln v_1, \ldots, \ln v_n$ of order k $\sigma_k(\lambda) = \sigma_k(\ln v_1, \ldots, \ln v_n)$.

The albebraic singular points λ_0 of the functions $v_1, \ldots, v_n$ will not be singular points of $\sigma_k(\lambda)$, since in the neighborhood of the point λ_0, the functions $\sigma_k(\lambda)$ are single-valued by virtue of being symmetric functions of $\ln v_1, \ldots, \ln v_n$, and by virtue of $v_k(\lambda) \neq 0$ in the region (1.2). This proves the theorem.

*Corollary**. If the series (1.1) is integral, then also the invariants of the matrix W will be integral functions of λ (with the condition $v_n(\lambda) \neq 0$).

In fact this Theorem was already proved in article [3] and has been mentioned in article [4].

2. Let the system of linear differential equations

$$dX / dt = XP(t) \qquad (2.1)$$

be given, where $P(t)$ is a continuous periodic matrix of order n with period $\omega = 2\pi$ in the region $t \geqslant 0$, and X an integral matrix of order n. The matrix $X(t)$ normalized at the point $t = 0$ can be represented in the form

$$X(t) = e^{At} Z(t) \qquad (2.2)$$

where A is a constant n-order matrix relative to t and $Z(t)$ is periodic with period $\omega = 2\pi$.

$$Z(0) = I, \qquad 2\pi A = \ln X(2\pi) \qquad (2.3)$$

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* Clearly the theorem and the corollary hold for any function $Y = f(V(\lambda))$, where $f(z)$ is analytic in the region (1.2) and where the eigenvalues of the matrix $V(\lambda)$ do not take on singular values of the function $f(z)$.

We shall here take $\ln X(2\pi)$ as the principal value. We will assume that among the characteristic roots of the matrix $X(2\pi)$ there are negative roots [4]. Then $Z(t)$ will have period 4π if we insist that A be real. But in the last case A is not determined [4] by the equality (2.3).

3. Let the linear system of differential equations

$$dX/dt = XP(t, \lambda) \tag{3.1}$$

be given, where the n-order matrix

$$P(t, \lambda) = \sum_{k=0}^{\infty} P_k(t)\lambda^k \tag{3.2}$$

the numerical parameter λ, and the matrices $P_k(t)$ are continuous and periodic with period 2π. The series (3.2) converges in the region $|\lambda| < r$.

The integral matrix $X(t, \lambda)$ normalized at $t = 0$, has, according to formula (2.2), the following representation:

$$X(t, \lambda) = e^{A(\lambda)t} Z(t, \lambda), \qquad Z(0, \lambda) = I \tag{3.3}$$

$$X(t, \lambda) = \sum_{k=0}^{\infty} X_k(t)\lambda^k \tag{3.4}$$

Here the series (3.4) converges in the region $|\lambda| < r$. As $\lambda \to 0$ we will have $X(t, \lambda) \to X_0(t)$ where

$$\frac{dX_0(t)}{dt} = X_0(t) P_0(t) \tag{3.5}$$

By formula (2.2)

$$X_0(t) = e^{A_0 t} Z_0(t) \tag{3.6}$$

Let the characteristic values of the matrix $X_0(2\pi)$ be $x_1^0, \ldots, x_n^0$ and the characteristic values of the matrix $2\pi A_0$ be the principal values $\ln x_1^0, \ldots, \ln x_n^0$ with none of the x_i^0 negative.

When $x_i^0 = x_k^0$, also $\ln x_i^0 = \ln x_k^0$, and the matrix $2\pi A_0 = \ln X_0(2\pi)$ will be real [4]. Then [4]

$$\ln X(2\pi, \lambda) = 2\pi A(\lambda) \to 2\pi A_0 \quad \text{for } \lambda \to 0$$

and hence*

$$A(\lambda) = \frac{1}{2\pi} \ln X(2\pi, \lambda) = A_0 + \sum_{k=1}^{\infty} A_k \lambda^k \tag{3.7}$$

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* We will note that in article [5] the discussion centers about the expansion of the canonical form $A(\lambda)$; in article [6] Lemmas 1.1 and 1.2, Theorem 2.2, and the formula for $Y^n(r, \epsilon)$ (p.30) repeat the known results.

Here the series (3.7) converges in the region $|\lambda| < r_1 < r$, but the invariants of the matrix $A(r)$ are, in accordance with Theorem 1, holomorphic in the region $|\lambda| < r$. Moreover $Z(t + 2\pi, \lambda) = Z(t, \lambda)$, and

$$Z(t, \lambda) = \sum_{k=0}^{\infty} Z_k(t)\lambda^k \tag{3.8}$$

If some of the x_i^0 are negative, we have

$$2\pi A_0 = A_1^\circ + \pi i A_2^\circ = \ln X_0(2\pi)$$

where A_2^0 is a matrix commuting with A_1^0, which has a diagonal canonical form, such that the eigenvalues are equal to zero, if their position corresponds to the non-negative eigenvalues x_i^0, and equal to one, if their position corresponds to the negative x_i^0. The matrix A_1^0 is real. In this case we can write [4]

$$X_0(t) = \exp\left[\frac{1}{2\pi} \ln X_0(2\pi)\right] Z_0(t) = \exp\left[\frac{1}{2\pi} A_1^\circ t\right] Z_1(t)$$

$$Z_1(t) = \exp\left[\frac{i}{2} A_2^\circ t\right] Z_0(t) \tag{3.9}$$

Hence the matrix $Z_1(t)$ will have a period 4π. Here

$$X(4\pi, \lambda) \to X_0(4\pi, 0) \quad \text{when } \lambda \to 0$$

$$\frac{1}{4\pi} \ln X(4\pi, \lambda) \to \frac{1}{4\pi} \ln X_0(4\pi, 0) = \frac{1}{2\pi} A_1^\circ \quad \text{for } \lambda \to 0$$

The expression $1/4\pi \ln X(t, \lambda)$ can be represented in the form

$$\frac{1}{4\pi} \ln X(t, \lambda) = \frac{1}{2\pi} A_1^\circ + \sum_{k=1}^{\infty} A_k \lambda^k = A(\lambda)$$

In all the cases, whenever in formula (3.3) the matrix $A(\lambda)$ has a series representation in powers of λ, the matrix $Z(t, \lambda)$ can also be represented in such a series. This can be seen from (3.3) where the free term of such a series will be $Z_0(t)$ or $Z_1(t)$ (from (3.9)).

A special case of the system (3.1) is the case when $P_0(t) = P_0$ is a constant matrix.

It follows that if the characteristic values p_l^0 of the matrix P_0 are such that $p_l^0 - p_k^0 \neq im$ (m integral), then we must take in formula (3.6)

$$A_0 = P_0, \qquad Z_0(t) = I$$

Now let some of the p_l^0 have the form

$$p_l^\circ = \mu_l^\circ + \frac{1}{2} m_l i, \quad |\mathrm{Im}(\mu_l - \mu_k)| < \frac{1}{2} \quad (m \text{ is an integer})$$

Then we can write

$$2\pi P_0 = \bar{A}_1 + \pi i \bar{A}_2, \qquad \bar{A}_1\bar{A}_2 = \bar{A}_2\bar{A}_1, \qquad |\mathrm{Im}(\bar{a}_{1l} - \bar{a}_{1k})| < \pi$$

and the characteristic values of the matrix A_2, which has a diagonal canonical form, will be zeros or integers; $\bar{a}_{1k} - x - r$ of the matrix $\bar{A}_1$.

We can write

$$X_0(t) = e^{P_0 t} = \exp\left(\frac{1}{2\pi}\bar{A}_1 t\right)\exp\left(\frac{1}{2}\bar{A}_2 it\right) \tag{3.10}$$

and let in (3.6)

$$A_0 = \frac{1}{2\pi}\bar{A}_1,\ Z_0(t) = \exp\left(\frac{1}{2}\bar{A}_2 it\right)$$

Now we will have

$$X(2\pi, \lambda) \to X_0(2\pi, 0), \qquad \frac{1}{2\pi}\ln X(2\pi, \lambda) \to A_0 = \frac{1}{2\pi}\bar{A}_1$$

$$Z(t, \lambda) \to z_0(t) = \exp\left(\frac{1}{2}\ A_2 it\right)$$

as $\lambda \to 0$ also (3.7), (3.8) hold if m_l are even.

If among the characteristic values $p_l^0 = p_l^0 + (m/2)i$ of the matrix P_0 there are such values that m is odd, then $Z_0(t) = \exp 1/3\, A_2 it$ will be the period 4π. Then also $Z(\lambda, t)$ has period 4π. Thus we will have (3.3), where

$$A(\lambda) = \frac{1}{2\pi}\bar{A}_1 + \sum_{k=1}^{\infty} A_k\lambda^k, \qquad Z(t, \lambda) = \exp\left(\frac{1}{2}\bar{A}_2 it\right) + \sum_{k=1}^{\infty} Z_k(t)\lambda^k$$

A method for the determination of the coefficients of the expansion (3.8) is given* in article [4].

We may proceed somewhat differently [5]. In particular, supposing $P_0(t) = P_0$ to be a constant matrix, we can replace the integral matrix X by the matrix Y, using** the equality

$$X = Y\exp\left(\frac{1}{2}\bar{A}_2 it\right) \tag{3.11}$$

Substituting in (3.2), we shall obtain an equation for the determination of

$$\frac{dY}{dt} = Y\left[P_0 + \sum_{k=1}^{\infty} P_k(t)\lambda^k - \frac{i}{2}\bar{A}_2\right] = Y\left[\frac{1}{2\pi}\bar{A}_1 + \sum_{k=1}^{\infty} P_k(t)\lambda^k\right]$$

Now we shall find

$$Y = \exp\left(\frac{1}{2\pi}\bar{A}_1 + \sum_{k=1}^{\infty} A_k\lambda^k\right)t\left(I + \sum_{k=1}^{\infty} \bar{Y}_k(t)\lambda^k\right)$$

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* This method has been formulated in article [3].

** See article [3] p. 10.

and consequently,

$$X = \exp\left(\frac{1}{2\pi}\bar{A}_1 + \sum_{k=1}^{\infty} A_k\lambda^k\right)t\left(\exp\frac{1}{2}\bar{A}_2 it + \sum_{k=1}^{\infty} Y_k(t)\lambda^k\right)$$

In other words, the operation of replacement of the variable in (3.11), in the case of the constant matrix P_0, is equivalent to the foregoing operations, where we set

$$A_0 = \frac{1}{2\pi}\bar{A}_1, \qquad Z_0 = \exp\frac{1}{2}\bar{A}_2 it$$

It must be noted, however, that in the case of the system (3.1) (regardless of whether $P_0(t)$ is constant or not) the method for the determination of the coefficients of the expansions (3.7) and (3.8) is very unwieldy. We offer, in the following Sections, another method for special cases.

4. Let us consider a system (3.1) in which $P_0(t), \ldots, P_m(t)$ are constant matrices. We shall further assume that the matrix P_0 does not have* characteristic values p_k, p_l such that $p_k - p_l = mi$ (m integral).

We shall seek a solution of the system (3.1) in the form (3.3), where $A(\lambda)$ and $Z(t, \lambda)$ have the forms (3.7) and (3.8) respectively. Moreover, $A_0 = P_0$ and $Z_0 = 1$.

We have, for the determination of Z_k and A_k, the equation

$$\frac{dZ_k}{dt} = \sum_{l=0}^{k} Z_{k-l}P_l - \sum_{l=0}^{k} A_{k-l}Z_l \tag{4.1}$$

We must look for periodic $Z_k(t)$ and $Z_k(0) = 0$. We see from (4.1)

$$Z_k = 0, \qquad A_k = P_k \qquad (k = 1, \ldots, m)$$

Thus we have

$$A(\lambda) = P_0 + P_1\lambda + \cdots + P_m\lambda^m + A_{m+1}\lambda^{m+1} + \cdots \tag{4.2}$$

$$Z(t, \lambda) = I + \sum_{k=m+1}^{\infty} Z_k(t)\lambda^k \tag{4.3}$$

We may set approximately

$$A(\lambda) \sim P_0 + P_1\lambda + \cdots + P_m\lambda^m \tag{4.4}$$

In finding the characteristic values of the matrix $A(\lambda)$ by means of the equality (4.4), we can use, by Theorem 1, the convergence of the series (3.2) in the region $|\lambda| < r$.

We may also proceed as follows: Writing the system (3.1) in the form

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* We can always make such a reduction, as we have just seen.

$$\frac{dx}{dt} = x[P(\lambda) + P_1(t, \lambda)] \tag{4.5}$$

where

$$P(\lambda) = \sum_{k=0}^{n} P_k \lambda^k, \qquad P_1(t, \lambda) = \sum_{k=m+1}^{\infty} P_k(t) \lambda^k$$

We shall consider the auxiliary system

$$\frac{dx}{dt} = X[P(\lambda) + P_1(t, \lambda)]\varepsilon \quad (\epsilon \text{ is a parameter}) \tag{4.6}$$

We shall now seek a solution of the system (4.6) in the form

$$X = \exp A(\varepsilon)t \cdot Z(t, \varepsilon) \tag{4.7}$$

where

$$A(\varepsilon) = \sum_{k=1}^{\infty} A_k \varepsilon^k, \qquad Z(t, \varepsilon) = I + \sum_{k=1}^{\infty} Z_k(t) \varepsilon^k \tag{4.8}$$

In this form the solution always exists [3,4] for ϵ sufficiently small.

We have for the determination of Z_k and A_k the equation

$$\frac{dZ_k}{dt} = Z_{k-1}[P(\lambda) + P_1(t, \lambda)] - A_k - \sum_{l=1}^{k-1} A_l Z_{k-l} \tag{4.9}$$

$$\frac{dZ_1}{dt} = P(\lambda) + P_1(t, \lambda) - A_1 \tag{4.10}$$

Hence

$$A_1 = \frac{1}{2\pi}\int_0^{2\pi} P_1(t, \lambda)\, dt + P(\lambda), \qquad Z_1 = \int_0^t P_1(t, \lambda)\, dt - \frac{1}{2\pi} t \int_0^{2\pi} P_1(t, \lambda)\, dt \tag{4.11}$$

In finding the characteristic values of the matrix $A(\epsilon)$ we can put $\epsilon = 1$, since the invariants of the matrix $A(\epsilon)$ are, by Theorem 1, integral functions of ϵ. Moreover we see from (4.9) and (4.11) that $Z_1(t)$, $Z_2(t)$ ..., as well as A_2, A_3 ... are small, of order $m + 1$ relative to λ.

Therefore to within magnitudes of order $m + 1$, the characteristic values of the matrix A(1) of the system of differential equations (3.1) can be sought, in this case, by setting

$$A = P(\lambda) \tag{4.12}$$

We have obtained the previous result; here however we may consider $P_1(t, \lambda)$ only as small, of order $m + 1$, without presupposing analyticity with respect to λ.

5. Let us again consider the system (3.1), this time assuming $P_0(t) = P_0$ to be constant with the property $p_k - p_l \neq mi$. In this case the solution

$X(t)$ can be represented in the form (3.3) where $A(\lambda)$ and $Z(t, \lambda)$ have the series representation (3.7) and (3.8). This solution, however, can be written somewhat differently. We shall apply the abridged Krylov-Bogoliubov transformation method, which was used by Shtokalo [8] and also by the author [9]. First we obtain

$$X = \zeta\Big[I + \sum_{k=1}^{m} \varepsilon^k Z_k(t)\Big], \qquad \frac{d\zeta}{dt} = \zeta\Big[\sum_{k=0}^{m} A_k\lambda^k + \lambda^{m+1}R_m(\lambda, t)\Big] \tag{5.1}$$

and then, in accordance with Section 4

$$\zeta = \exp\Big(\sum_{k=0}^{\infty} A_k\lambda^k t\Big)Z(t,\lambda), \qquad Z(t,\lambda) = I + \sum_{k=1}^{\infty} Z_k(t)\lambda^k \tag{5.2}$$

where $A_0 = P_0$, and A_k, Z_k $(k = 1, \ldots, m)$ are determined from equations (4.1)

Thus

$$X = \exp\Big(\sum_{k=0}^{\infty} A_k\lambda^k t\Big)\Big[I + \sum_{k=m+1}^{\infty} Z_k(t)\lambda^k\Big]\Big[I + \sum_{k=1}^{m} Z_k(t)\Big] \tag{5.3}$$

If the condition $p_k - p_l \neq mi$ is not satisfied, then, as was shown in the work of Shtokalo [8], the characteristic values of the matrix A_0 in the second equation of (5.1) will satisfy such a condition, while in the first equation we shall have, instead of the matrix I, $B \exp Vit$, where B is a constant matrix and V a diagonal matrix whose elements are integers.

In article [8], Shtokalo considers a system of the form

$$\frac{dX}{dt} = X\,[A + \varepsilon f(t)] \qquad \Big(F(t) = \sum_{\mu} C_\mu e^{i\mu t}\Big)$$

where $F(t)$ is a finite sum. But clearly nothing is changed in the method if we consider a system of the form

$$\frac{dx}{dt} = XP(t, \varepsilon) \qquad \Big(P(t,\varepsilon) = A + \sum_{k=1}^{\infty} P_k(t)\,\varepsilon^k\Big) \tag{5.4}$$

where $P_k(t)$ $(k = 1, \ldots, m)$ are either matrices of the form $F(t)$ or periodic.

We must only make a reduction to the case $a_k - a_l \neq mi$ (where a_k are the characteristic values of the matrix A and m is integral) and find A_k, Z_k by the method of Section 8, article [4 [, or by the method of article [8].

We may assume $P_k(t)$ to be uniformly quasiperiodic functions [10] with exponents* $\Lambda_l^{(k)} \to \infty$ as $l \to \infty$ and also with certain other exponents. The

* For simplicity we assume the totality $\{\Lambda_l^{(k)}\}_{l=1}^{\infty}$ of the exponents of the matrices $P_k(t)$ to be independent of k.

matrices A_k and Z_k ($k = 1, \ldots, m$) can be found in many cases. (See for example article [3]. But since Theorem 1 does not hold in these cases, the stability problem of the solutions must be investigated by Shtokalo's method, i.e. by the construction of Liapunov's function. Consequently, the matrices

$$\sum_{k=0}^{m} A_k \lambda^k$$

cannot be considered* as approximate values of the exponents.

If $P_0(t)$ is not a constant matrix, then we can use the abridged transformation

$$X = \zeta\Big[Z_0(t) + \sum_{k=1}^{m} Z_k(t)\lambda^k\Big], \qquad \frac{d\zeta}{dt} = \zeta\Big[\sum_{k=0}^{m} A_k\lambda^k + \lambda^{m+1}R_m(\lambda, t)\Big] \quad (5.5)$$

where $Z_0(t)$, A_0 are determined from equations (3.5) and (3.6) and A_k, $Z_k(t)$ ($k = 1, \ldots, m$) are found from the equations (4.1):

$$\frac{dZ_k}{dt} = Z_k P_0 - A_0 Z_k + \sum_{l=2}^{k} Z_{k-l} P_k - \sum_{l=0}^{k-1} A_{k-l} Z_l \quad (5.6)$$

with $Z_k(t)$ subjected to the periodicity condition, and $Z_k(0) = 0$.

Section 8 of article [2] gives formulas for the determination of $Z_k(t)$ and A_k [(8.45), (8.46)]. Next, if $R_m(\lambda, t)$ is a periodic matrix, ζ can be sought in (5.5) by the method of Section 4 of this article. If in (3.6) the characteristic values A_0 are pure imaginaries and simple, we may proceed in a different manner.

Replacing

$$X = YX_0 \quad (5.7)$$

we obtain

$$\frac{dY}{dt} = Y\,[X_0 P X_0^{-1} - X_0 P_0 X_0^{-1}] = Y\,[X_0 P_1(t, \lambda) X_0^{-1}] \quad (5.8)$$

where

$$P_1(t, \lambda) = P - P_0 = \sum_{k=1}^{\infty} P_k(t)\lambda^k$$

Substituting in (5.8) the expression X_0 from (3.6) we will find

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* Provided we are not working with a system for which the stability of the eigenvalues is valid. (In this connection see the works of B.F. Bylov, I.G. Malkin, R.E. Vinogradov, Iu.S. Bogdanov).

$$\frac{dY}{dt} = Y\,[e^{A_0 t} Z_0(t)\, P_1(t, \lambda)\, Z_0^{-1}(t)\, e^{-A_0 t}] = YQ(t, \lambda) \tag{5.9}$$

Here the coefficient matrix $Q(t, \lambda)$ has the form

$$Q(t, \lambda) = \sum_{k=1}^{\infty} Q_k(t)\, \lambda^k \tag{5.10}$$

where the elements of the matrices $Q_k(t)$ will be periodic if the matrix $\exp(A_0 t)$ is periodic, and of such period as $P_k(t)$, or if it has the form $p(t)$ exp (ait), with the function $p(t)$ periodic and a real (if A_0 has simple elementary divisors). Hence we can apply to the system (5.9) Shtokalov's method [8]. Here we shall get for the determination of A_k equations (4.10), where we must put $P_0 = A_0 = 0$. Hence the matrices Z_k and A_k will be easily determined (since $P_0 = A_0 = 0$, see (5.6)) from the condition of boundedness for $Z_k(t)$ (since the function $p(t)$ exp (iat) has a mean value [10] or from the periodicity of $Z_k(t)$, provided the $Q_k(t)$ are periodic. If in (5.10) the matrices $Q_k(t)$ turn out to be periodic, then we may take for approximate values of the exponents of the matrix Y the eigenvalues of the matrix

$$A_1\lambda + A_2\lambda^2 + \ldots + A_m\lambda^m$$

We will note in conclusion that if the matrices $P_k(t)$ are periodic with a single period 2π, then by (3.4) we can find the matrix (3.7) directly (without finding $Z_k(t)$), using the formula

$$A(\lambda) = \frac{1}{2\pi} \ln X(2\pi, \lambda)$$

Here we take always the principal value of the logarithm and use the formulas (1.31), (1.32) of article [2], or the corresponding more general formula for n-order matrices of article [12]. This will absolve us from examining the special cases when the values of $\ln X_0(2\pi)$ are regular or irregular.

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Translated by I.S.